AD-A079 642 JOHNS HOPKINS UNIV LAUREL MD APPLIED PHYSICS LAB F/G 20/4
THE FORMULATION OF SELECTED FREE BOUNDARY PROBLEMS AS CONSERVAT--ETC(U)
DEC 79 J C ROGERS N00014-79-RC-90009
UNCLASSIFIED APL/JHU REPRINT-2173 NL

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

PLEASE FOLD BACK IF NOT NEEDED FOR BIBLIOGRAPHIC PURPOSES

# REPORT DOCUMENTATION PAGE

LEVEL II

| | |
|---|---|
| 1. REPORT NUMBER: APL/JHU Reprint [illegible] 2173 | 2. GOVT ACCESSION NO. |
| 3. RECIPIENT'S CATALOG NUMBER | |
| 4. TITLE (*and Subtitle*): The Formulation of Selected Free Boundary Problems As Conservation Laws | 5. TYPE OF REPORT & PERIOD COVERED: Reprint |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s): Joel C.W. Rogers | 8. CONTRACT OR GRANT NUMBER(s): N00014-79-RC-9[illegible]9 |
| 9. PERFORMING ORGANIZATION NAME & ADDRESS: The Johns Hopkins University Applied Physics Laboratory, Johns Hopkins Road, Laurel, MD 20810 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: ONR-NR-334-003 |
| 11. CONTROLLING OFFICE NAME & ADDRESS: Office of Naval Research, 800 North Quincy Street, Arlington, VA 22217 | 12. REPORT DATE: December 1979 |
| | 13. NUMBER OF PAGES |
| 14. MONITORING AGENCY NAME & ADDRESS: Naval Plant Representative, Laurel, MD; Applied Physics Laboratory, Johns Hopkins Road, Laurel, MD 20810 | 15. SECURITY CLASS. (*of this report*): Unclassified |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE: None |

16. DISTRIBUTION STATEMENT (*of this Report*)

A

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

A

18. SUPPLEMENTARY NOTES

None

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

Conservation law
free boundary problem
Burgers' equation
hydrodynamics
Stefan problem
semi-groups
systems

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

Some free boundary problems are written as conservation laws. Numerical algorithms conserving the necessary quantities are discussed. Suitable function spaces are mentioned. Formal extensions of the algorithms to systems are proposed. Analytic difficulties in the solution of systems in higher spatial dimensions are suggested.



DD FORM 1 JAN 73 1473

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

C31 6-C

# THE FORMULATION OF SELECTED FREE BOUNDARY PROBLEMS AS CONSERVATION LAWS

Joel C. W. Rogers
Laurel, Maryland, U.S.A.

RE: Contract N00014-79-RC-90009
Contract is valid. It is a special in house naval contract per Ms. Hassan, ONR/Code 438



Reprinted from the Proceedings of the Intensive Seminar on Free Boundary Problems, Pavia, Italy, Sept.-Oct. 1979

# THE FORMULATION OF SELECTED FREE BOUNDARY PROBLEMS AS CONSERVATION LAWS

Joel C. W. Rogers
Laurel, Maryland, U.S.A.

## 1. Introduction and Examples

In this brief paper, we shall summarize some recent work on free boundary problems which may also be considered as conservation laws. This is by no means a review article, and at best it is a survey of currents of thought which have emerged in work in which the author has participated, mostly in collaboration with others. This may explain the rather parochial nature of the list of references at the end. The present state of our knowledge is quite rudimentary, and it will be apparent to the reader that conjectures considerably outnumber theorems. The subject matter of this paper will be fairly broad; the price of this breadth will be a lack of depth.

Generally speaking, a set of differential equations will be said to be in conservation form if, by formal integration of the equations over a region, one can obtain equations for the rate of change of the dependent variables integrated over the region in terms of their values on the boundary of the region and other quantities which are specified for the problem.

A typical example is the one-phase Stefan problem. Choosing units so that the latent heat is 1 and the melting temperature is 0, one may

formulate the problem classically as the solution of

$$u_t = \Delta u \quad , \quad x \in \Omega(t) \quad , \quad t > 0 \quad , \tag{1.1a}$$

$$u(x,t) \xrightarrow[x\in\Omega(t)\to\partial\Omega(t)]{} 1 \quad , \quad t > 0 \quad , \tag{1.1b}$$

$$u(x,t) = 0 \quad , \quad x \notin \Omega(t) \quad , \quad t > 0 \quad , \tag{1.1c}$$

$$u(x,t) \xrightarrow[t\to 0]{} u_0(x) \quad , \quad x \in R^N \quad , \tag{1.1d}$$

where u is the energy per unit volume and

$$u_0(x) \geq 1 \quad , \quad x \in \Omega(0) \quad , \tag{1.1e}$$

$$u_0(x) = 0 \quad , \quad x \notin \Omega(0) \quad . \tag{1.1f}$$

To complete the statement of the problem, one must indicate how the region $\Omega(t)$ changes with time. The rate of change of $\Omega$ is determined by requiring conservation of energy and noting that, with the units chosen, the flux of energy into $\partial\Omega$ from the interior of $\Omega$ is $-\frac{\partial u}{\partial n}$, and this must equal the flux of material volume which has changed phase times the latent heat. Thus, if v is the outward normal velocity of the boundary $\partial\Omega(t)$,

$$v = -\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega(t)} \quad . \tag{1.1g}$$

Since conservation of energy was invoked to complete the formulation of problem (1.1), there is a certain logic to writing the problem directly as a conservation law:

$$u_t = \Delta f(u) \quad , \tag{1.2a}$$

$$f(u) = \max(u - 1, 0) \quad , \tag{1.2b}$$

$$u(t) \xrightarrow[t\to 0]{} u_0 \quad . \tag{1.2c}$$

Using Gauss's theorem, we see from (1.1c) that, if $\Omega$ is bounded, $\int u dx$ is conserved. It is easy to check that the left-hand side of (1.2a) has a singular part $v\delta_{\partial\Omega}$, and the right-hand side has a singular part $-\frac{\partial u}{\partial n}\delta_{\partial\Omega}$,

where $\delta_{\partial\Omega}$ is a Dirac measure on $\partial\Omega$. Equating the two singular parts, we recover (1.1g).

An advantage of the formulation (1.2) over (1.1) is that in (1.2) the topology of the set $\{x|u(x) \geq 1\}$ does not appear, whereas (1.1g) is meaningless if n is not defined on $\partial\Omega(t)$ or $\frac{\partial u}{\partial n}$ is not bounded. A more subtle advantage is that, by formulating (1.2) as the set of equations (1.1), we are already making an assumption about the nature of the solution, and such an assumption may be unwarranted. In (1.1), we assume that u has a jump from 1 to 0 at the free boundary, that is, that the front between $\{x|u(x) \geq 1\}$ and $\{x|u(x) = 0\}$ is sharp. However, in the inhomogeneous problem

$$u_t = \Delta f(u) + \sigma(x) \tag{1.3a}$$

where $\sigma(x) \geq 0$ is bounded and

$$\text{supp } \sigma \cap \text{supp } u_0 = \emptyset \quad , \tag{1.3b}$$

there will not be a sharp front. Instead, for a period of time there will be a diffuse boundary between $\{x|u(x,t) \geq 1\}$ and $\{x|u(x,t) = 0\}$, and any attempt to state the problem in the form (1.1) will be doomed to failure.

A different example is afforded by the hyperbolic conservation law

$$u_t + \nabla \cdot (f(u)) = 0 \quad , \tag{1.4a}$$

$$u(x,t) \xrightarrow[t\downarrow 0]{} u_0(x) \quad . \tag{1.4b}$$

Generally, solutions of this equation will not be continuous. For discontinuous solutions the singular parts of the terms on the left-hand side of (1.4) will have to cancel one another. Define

$$\pm u^{\pm}(x,t) \equiv \lim \sup_{y \to x} \pm u(y,t) \tag{1.5}$$

and

$$\partial\Omega(t) \equiv \{x|u^{+}(x,t) - u^{-}(x,t) > 0\} \quad . \tag{1.6}$$

Then the balancing of singularities gives for the velocity v of $\partial\Omega(t)$,

$$v(u^+(x,t) - u^-(x,t)) = f(u^+(x,t)) - f(u^-(x,t)). \qquad (1.7)$$

A given differential equation corresponds to a multitude of conservation laws, and the solution of the equation which is considered acceptable is critically dependent on the quantity assumed to be conserved. For example, the equation

$$u_t + uu_x = 0 \qquad (1.8a)$$

may be written as

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad , \qquad (1.8b)$$

or

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0 \quad . \qquad (1.8c)$$

In (1.8b) the conserved quantity is $\int u dx$. In (1.8c) it is $\int u^2 dx$. The general solutions of the two conservation laws are quite different.

## 2. Numerical Solution of Some Simple Conservation Laws

One naturally looks for algorithms which rigorously preserve the required invariants of the conservation laws, and in addition have the feature that they generate sufficiently smooth solutions of the conservation laws with satisfactory accuracy. For the hyperbolic conservation law (1.4), such an algorithm has been studied (Ref. 5). To explain how the algorithm works, we introduce the sets

$$E_u(\alpha) \equiv \{x \,|\, u(x) > \alpha\} \quad , \quad \alpha > 0 \quad , \qquad (2.1a)$$

$$E_u(\alpha) \equiv \{x \,|\, u(x) < \alpha\} \quad , \quad \alpha < 0 \quad . \qquad (2.1b)$$

In addition, for $1 \leq i \leq N$, we define the operators $S_i(h)$ by

$$(S_i(h)u)(x) = u(x - he_i) \quad , \qquad (2.2)$$

where $e_i$ is a unit vector in the direction of the $x_i$-axis. The algorithm generates a collection of functions $u^n(x)$, $n = 0, 1, 2, \cdots$, dependent on a parameter $\tau$, called the time step. The purpose is to approximate $u(x,t)$ by $u^n(x)$ for $\tau = t/n$ sufficiently small. $u^n(x)$ is given by

$$u^n(x) = G^n u_0(x) \tag{2.3a}$$

where

$$Gu \equiv \int_0^\infty \prod_{i=1}^N S_i(f'(\alpha) \cdot e_i \tau) \chi(E_u(\alpha)) d\alpha$$

$$- \int_{-\infty}^0 \prod_{i=1}^N S_i(f'(\alpha) \cdot e_i \tau) \chi(E_u(\alpha)) d\alpha \quad . \tag{2.3b}$$

Here $\chi(E)$ is the characteristic function of the set E. For a problem in one space dimension, under mild regularity assumptions on $u_0$ and some assumptions on f (e.g., f convex), the algorithm (2.3) has been shown to generate an approximate solution with an $L^1$ error $O(\tau)$ (Ref. 5).

A numerical approach to a more general class of conservation laws which includes (1.2) and the one-dimensional version of (1.4) has been presented (Ref. 3). One considers the problem

$$u_t + Lf(u) = 0 \quad , \tag{2.4a}$$

$$u(x,t) \xrightarrow[t \downarrow 0]{} u_0(x) \quad , \tag{2.4b}$$

where

$$0 \leq f(u) - f(v) \leq u - v \text{ if } u \geq v \tag{2.4c}$$

and L is a linear operator such that the semi-group

$$S(h) = e^{-hL} \text{ is contractive in } L^1 \text{ and } L^\infty \quad . \tag{2.4d}$$

The numerical method generates functions $u^n(x)$, supposed to approximate $u(x,n\tau)$, according to the prescription

$$u^n(x) = F^n u_0(x) \quad , \tag{2.5a}$$

$$Fu \equiv u - f(u) + S(\tau)f(u) \quad . \tag{2.5b}$$

The functions $u^n(x)$ have been shown to converge to $u(x,t)$ as $\tau = t/n \to 0$ (Ref. 3). For the special case of problem (1.2) in one space dimension, the $L^1$ error in the approximate solution, under mild assumptions regarding the regularity of $u_0$, has been shown to be $0\left(\left(\tau \ \ell n \ \frac{1}{\tau}\right)^{\frac{1}{2}}\right)$ (Ref. 2).

Comparison with the procedure (2.3) suggests that there might be some theoretical advantage in solving (2.4) by considering the algorithm

$$u^n(x) = G^n u_0(x) \tag{2.6a}$$

where

$$Gu(x) \equiv \int_0^\infty S(f'(\alpha)\tau)\chi(E_u(\alpha))d\alpha - \int_{-\infty}^0 S(f'(\alpha)\tau)\chi(E_u(\alpha))d\alpha \quad . \tag{2.6b}$$

At this point, the convergence of $u^n(x)$ in (2.6a) to $u(x,t)$ as $\tau = t/n \to 0$ is only conjectured. It may be of interest to note that, upon use of the interpolation formula

$$S(f'(\alpha)\tau) \to 1 + f'(\alpha)(S(\tau) - 1) \quad , \tag{2.7}$$

(2.6) goes over to (2.5).

## 3. Function Spaces Approximate to Free Boundary Problems

A conservation law may strongly suggest a natural function space in which to search for a solution of the associated free boundary problem. This is especially true when the conserved quantity can be thought of as a norm for a large class of solutions. The space $L^1$ has already figured in error estimates for approximate solutions of (1.2) and (1.4).

In particular, for the conservation law (2.4a), it follows from (2.4c) and (2.4d) that the operator F is contractive in $L^1$. Further, the

algorithm (2.5) is stable in both $L^1$ and $L^\infty$. In cases of importance, F is stable in still other function spaces. For example, if S(h) not only satisfies (2.4d), but has the property that

$$(S(h)u)(x) = \int G(x';h)u(x + x')dx' \quad , \tag{3.1}$$

one may find a whole class of function spaces in which F is stable. One of the most interesting is the space with the norm

$$||u|| = \int_{-\infty}^{\infty} |\partial E_u(\alpha)|d\alpha \quad . \tag{3.2}$$

Using this space and the algorithm (2.5), one may deduce immediately bounds on the measures of surfaces of discontinuity of solutions of Stefan problems and solutions of model hyperbolic free boundary problems in terms of the initial data. More generally, one may introduce function spaces whose norms are constructed in terms of capacities of sets $E_u(\alpha)$. Such spaces and others, and their use in studying the regularity of solutions of the conservation law (2.4), are treated in more detail elsewhere (Ref. 4).

## 4. The Formal Extension to Systems of Conservation Laws

One might consider u and f(u) to be n-vectors, in which case (2.4) becomes a system of n conservation laws. The numerical scheme (2.5) still makes sense for this case, and it is at least consistent with the differential equation. If L has the property (2.4d), and if in addition $\forall u \in R^n$ we have

$$|u - f(u)|_1 + |f(u)|_1 = |u|_1 \quad , \tag{4.1}$$

then we get stability of the operator F in $L^1$:

$$||u||_{L^1} = \int |u|_1 dx \quad . \tag{4.2}$$

However, the contractiveness of F in $L^1$, which followed so easily for the case of a single equation, does not necessarily follow for a system of equations. In addition it appears that, of all the spaces alluded to above, with norms constructed in terms of capacities of the $E_u(\alpha)$, the only space in which we can expect to get $||F|| \leq 1$, except for trivial cases, is $L^1$.

A second example is afforded by Burgers' equation in N dimensions, by which we mean the laws of mass and momentum conservation of a perfectly compressible inviscid fluid (no pressure). Burgers' equation is

$$\rho_t + \nabla \cdot (\rho u) = 0 \quad , \tag{4.3a}$$

$$(\rho u)_t + \nabla \cdot (\rho u u) = 0 \quad , \tag{4.3b}$$

$$\rho(x,t) \xrightarrow[t \downarrow 0]{} \rho_0(x) \quad , \tag{4.3c}$$

$$(\rho u)(x,t) \xrightarrow[t \downarrow 0]{} (\rho u)_0(x) \quad . \tag{4.3d}$$

A formal algorithm seeks the solution of (4.3) by means of a "Boltzmann equation" approach (Refs. 6, 8). The method is similar to the method of (2.3). One writes

$$\begin{pmatrix} \rho^n \\ (\rho u)^n \end{pmatrix} = G^n \begin{pmatrix} \rho_0 \\ (\rho u)_0 \end{pmatrix} \tag{4.4a}$$

where G is defined by

$$G\begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \begin{pmatrix} \int \left( \prod_{i=1}^{N} S_i(v \cdot e_i \tau) \right) (\rho \delta(v - u)) dv \\ \int v \left( \prod_{i=1}^{N} S_i(v \cdot e_i \tau) \right) (\rho \delta(v - u)) dv \end{pmatrix} \tag{4.4b}$$

and $S_i$ is given in (2.2). One can show that solutions of (4.3) for which u is sufficiently smooth, uniformly in time, are given exactly by (4.4).

However, from the point of view of free boundary problems, it is the nonsmooth solutions which are of primary interest.

A further formal extension of the methods described above would be to constrained systems of conservation laws. By these, we mean simply conservation laws for which the solution is required to lie in some subset of a function space, with the corresponding differential equations holding for solutions in the interior of that subset. For example, the equations of inviscid hydrodynamics may be described succinctly as the solution of Burgers' equation (4.3) subject to the one-sided density constraint $\rho \leq 1$ (Ref. 6). The corresponding free boundary problem is the water wave problem.

In a stepwise solution of a constrained conservation law, it is most natural to let each step consist of two procedures: First, the solution for a step of the corresponding unconstrained conservation law, as described heretofore, and then the satisfaction of the constraint. In the satisfaction of the constraint, any numerical procedure should carefully conserve the required invariants of the system. For example, in hydrodynamics the requirements imposed by the one-sided constraint on the density are met by solving another conservation law, in this case the one-phase Stefan problem (Refs. 6, 8).

Another example of a constrained conservation law is the "diffusion-consumption" problem (Ref. 1). Here one solves the conservation law

$$u_t = \Delta u - 1 \quad , \tag{4.5a}$$

$$u(x,t) \xrightarrow[t \downarrow 0]{} u_0(x) \quad , \tag{4.5b}$$

subject to the constraint

$$u(x,t) \geq 0 \quad . \tag{4.5c}$$

In some cases it may be fruitful to consider the specification of conditions on a function at the boundary of a finite region $\Omega \subset R^N$ as the imposition of a constraint on a conservation law which is formulated throughout $R^N$, and to solve the resulting constrained conservation law stepwise, in accordance with the procedure outlined above.

## 5. Problems Associated with Coupled Equations in Several Space Dimensions

The free boundary problems of mechanics which have elicited the greatest interest have involved coupled equations in several space dimensions. The simplest of these is the water wave problem described above. A slightly more difficult problem is the transonic flow problem, which can be considered to be given by the conservation laws

$$\rho_t + \nabla \cdot (\rho u) = 0 \quad , \tag{5.1a}$$

$$(\rho u)_t + \nabla \cdot (\rho u u) = - \nabla P(\rho) \quad , \tag{5.1b}$$

where $P(\rho)$ is specified. The free boundary here is called the sonic surface. A more elaborate system of conservation laws, in which another dependent variable (entropy) and an extra conservation law (for energy) are added, describes the flow of an inviscid compressible fluid, and the corresponding free boundaries are shock waves.

For each of these sets of conservation laws, which may be thought of as progressively enriched versions of Burgers' equation, similar questions as to the well-posedness of the initial value problem arise. We noted in the last section that, although we can write down formal algorithms for

"solving" these conservation laws, the variety of spaces in which we get stability of the algorithms, and thus strong regularity results, diminishes drastically on passing from a single conservation law to a system of conservation laws. That the proof of strong regularity results for such systems may not be just difficult, but in fact impossible, is strongly suggested by the irregularities actually observed in the physical world, and associated with the names of Helmholtz instability, Taylor instability, fingering of salt water-fresh water interfaces and oil-water interfaces, and turbulence. All these phenomena are associated with systems of equations in more than one space dimension. Since Burgers' equation is at the core of all these sets of conservation laws, it may not be unreasonable to expect limitations on the regularity of solutions of Burgers' equation to have some relevance to the solutions of the other systems. In the case of Burgers' equation, it would appear that the initial value problem is not well posed in the classical sense of Hadamard, and that it may be necessary to go to a concept of well-posing in a stochastic sense. In this case it may be that a given initial state will evolve stochastically after a finite time (Ref. 7).

Other systems of conservation laws in higher dimensions which are not built around Burgers' equation also appear to have highly irregular solutions. As a prototypical example, we might consider the system

$$u_t = \Delta f(u) \quad . \tag{5.2}$$

For example, equations like (5.2) have been used to describe phase changes in materials where the melting temperature depends on the concentration of a solute. In some of the models proposed, u has two components,

representing concentration of solute and energy density. It would appear that the free surfaces separating phases are highly unstable, and that this instability is reflected in the formation of highly irregular boundaries containing dendrites (Ref. 9).

Acknowledgment

This work has been supported by the Office of Naval Research under Task No. NR 334-003.

## References


1. Alan E. Berger, Melvyn Ciment, and Joel C. W. Rogers, "Numerical Solution of a Diffusion Consumption Problem with a Free Boundary," SIAM J. Numer. Anal. 12, 646 (1975).

2. Alan E. Berger, Melvyn Ciment, and Joel C. W. Rogers, "The Alternating Phase Truncation Method for Numerical Solution of a Stefan Problem," SIAM J. Numer. Anal. 16, 563 (1979).

3. Haim Brezis, Alan E. Berger, and Joel C. W. Rogers, "A Numerical Method for Solving the Problem $u_t - \Delta f(u) = 0$," to appear.

4. Haim Brezis, Alan E. Berger, and Joel C. W. Rogers, "A Numerical Method for Solving the Problem $u_t - \Delta f(u) = 0$, II," in preparation.

5. J. C. W. Rogers, "An Algorithm for a Hyperbolic Free Boundary Problem," The Johns Hopkins University Applied Physics Laboratory APL/JHU TG 1309 (1977).

6. Joel C. W. Rogers, "Incompressible Flows As a System of Conservation Laws with a Constraint," Seminaires IRIA, Analyse et Contrôle de Systèmes (1978).

7. Joel C. W. Rogers, "Stability, Energy Conservation, and Turbulence for Water Waves," Seminaires IRIA, Analyse et Contrôle de Systèmes (1978).

8. Joel C. W. Rogers and S. Favin, "Computation of Water Waves," The Johns Hopkins University Applied Physics Laboratory APL/JHU TG 1325 (1979).

9. R. F. Sekerka, "Morphological Stability," Crystal Growth: An Introduction, P. Hartman, Ed., 403 (1973).